Mechanical & Industrial Engineering
UNIVERSITY OF TORONTO

Integration of Support Vector Machines and Mean-Variance Optimization for Capital Allocation

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Preliminaries

Cardinality Constrained Portfolio Selection

- [1] was the first to propose selecting a portfolio of assets to optimize a trade-off between risk and expected return.
- Investors generally prefer portfolios with fewer assets (lower cardinality) due to frictions such as transaction costs, motivating the following optimization problem:

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^\intercal \boldsymbol{\Sigma} \boldsymbol{x} \quad s.t. \quad \mathbf{1}^\intercal \boldsymbol{x} = 1, \quad \boldsymbol{r}^\intercal \boldsymbol{x} \geq r_{\text{min}}, \quad |\text{supp}(\boldsymbol{x})| \leq K, \quad \boldsymbol{x} \geq 0 \tag{Card-MVO}$$

where
- A portfolio is represented by $\boldsymbol{x} \in \mathbb{R}^N_+$
- $\boldsymbol{r} \in \mathbb{R}^N$ denotes the estimated expectation of the random returns
- $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$ denote the covariance of the random returns
- r_{min} denotes the investor's minimum required expected return
- $K \in [N]$ representing a cardinality limit
- $|\text{supp}(\boldsymbol{x})|$ denotes the number assets receiving capital for investment

- Card-MVO admits a mixed integer quadratic formulation that is amenable to commercial solvers: Let $\boldsymbol{z} \in \{0,1\}^N$ denote binary variables such that $z_i = 0 \implies x_i = 0 \, \forall I \in [N]$.

Asset Screening Tools for Investment

- Investors prefer to have an asset screening tool that explains why a particular asset is or is not in the portfolio [2].
- Each asset i has a feature vector $\boldsymbol{y}^{(i)} \in \mathcal{Y} \subset \mathbb{R}^p$ where $\mathcal{Y}$ is a feature space defined by various asset attributes, such as price-earnings ratios and market capitalization.
- One can represent an asset screening tool by a hyperplane $H(\boldsymbol{w}, b) : = \{\boldsymbol{y} \mid \boldsymbol{w}^\intercal \boldsymbol{y} + b = 0\}$ where $(\boldsymbol{w}, b) \in \mathbb{R}^{p+1}$. The hyperplane acts as a screener and classifies asset i as eligible for investment if $\boldsymbol{w}^\intercal \boldsymbol{y} + b > 0$ and ineligible if $\boldsymbol{w}^\intercal \boldsymbol{y} + b < 0$.

Objective

There are two main drawbacks associated with the approaches mentioned above:

1. Card-MVO does not consider the structure of the assets in their associated feature space.
2. Data-driven approaches that construct hyperplane-based screeners do not consider the risk profile of the resulting portfolio in their estimation.

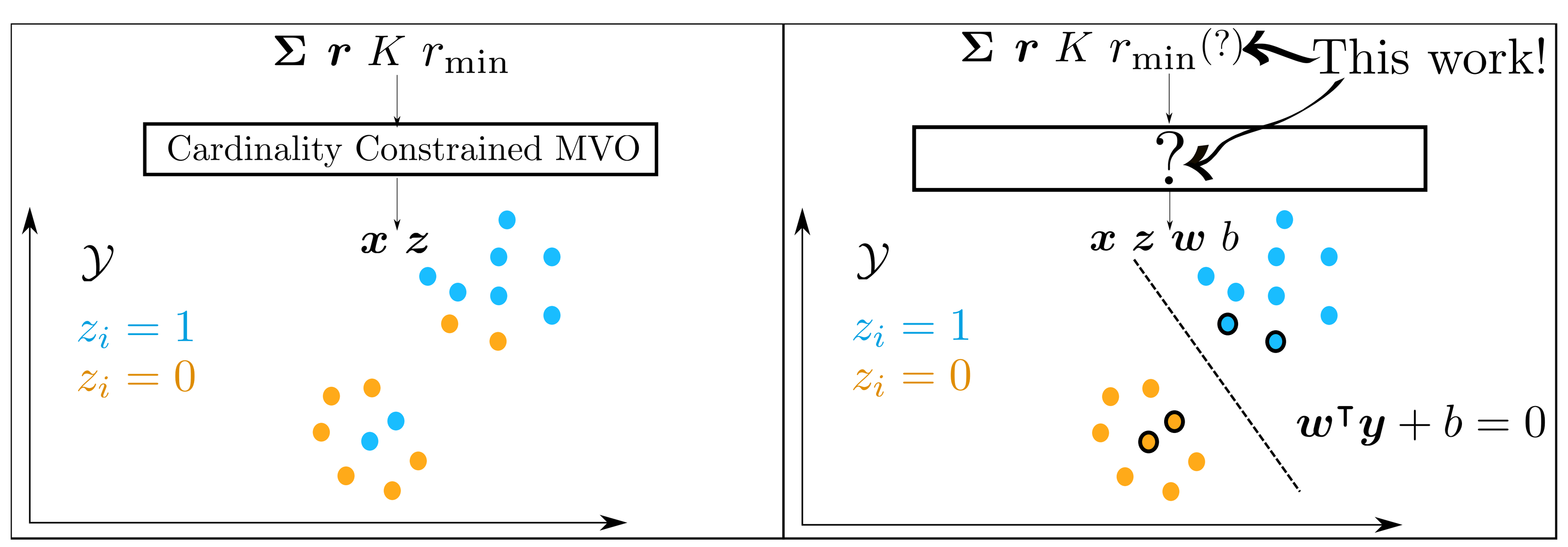

Figure 1. The desired outcome (right) shows how an asset screener could capture the structure of the feature space. In the desired outcome, the circled assets differ from the decision made by Card-MVO because they violate an investor's sense of eligibility based on an asset-screening hyperplane.

Objective: Address issues 1 and 2 and compare the proposed approach's financial performance with Card-MVO.

Mixed Integer Programming Approach

Model Rationale: jointly identify a portfolio $\boldsymbol{x}$ and hyperplane $H(\boldsymbol{w}, b)$ such that the portfolio has desirable risk-return properties and the hyperplane accurately classifies the assets selected by the optimizer.

- $H(\boldsymbol{w}, b)$ classifies assets as eligible ($z_i = 1$) or ineligible ($z_i = 0$).
- Binary variables $\boldsymbol{t} \in \{0,1\}^p$ are introduced such that $t_j = 0 \implies w_j = 0 \, \forall j \in [p]$ to ensure $q < p$ features are selected, resulting in the following optimization problem

$$\min_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{w},b,\boldsymbol{\xi},\boldsymbol{t}} \ \boldsymbol{x}^\intercal \boldsymbol{\Sigma} \boldsymbol{x} + \epsilon \left(\frac{1}{2} \|\boldsymbol{w}\|_2^2 + \frac{C}{N} \mathbf{1}^\intercal \boldsymbol{\xi} \right) \tag{SVM-MVO}$$

$$\begin{aligned}
\text{s.t.} \quad & -M(1 - z_i) + 1 - \xi_i \leq (\boldsymbol{y}^{(i)})^\intercal \boldsymbol{w} + b \leq M z_i - 1 + \xi_i, \quad \forall i \in [N] && (1)\\
& \boldsymbol{r}^\intercal \boldsymbol{x} \geq r_{\text{min}}, \ \mathbf{1}^\intercal \boldsymbol{z} \leq K, \ \boldsymbol{x} \leq \boldsymbol{z} && (2)\\
& \mathbf{1}^\intercal \boldsymbol{t} \leq q, \ -U_{\|\boldsymbol{w}\|_\infty} \boldsymbol{t} \leq \boldsymbol{w} \leq U_{\|\boldsymbol{w}\|_\infty} \boldsymbol{t} && (3)\\
& \boldsymbol{x} \in \Delta_N, \ \boldsymbol{z} \in \{0,1\}^N, \ (\boldsymbol{w}, b, \boldsymbol{\xi}) \in \mathbb{R}^{p+1} \times \mathbb{R}^N_+, \ \boldsymbol{t} \in \{0,1\}^p && (4)
\end{aligned}$$

where
- (1) - (3) model the hyperplane separation, portfolio and feature-selection constraints, and Δ_N is simplex,
- $U_{\|\boldsymbol{w}\|_\infty}$ and M are big-M constants, and C represents the relative preference for separation over margin, and ϵ weighs the asset-screening objective against the portfolio objective.

- SVM-MVO determines $H(\boldsymbol{w}, b)$ by solving the support vector machine (SVM) problem with features $\{y^{(i)}\}_{i=1}^N$ and labels $2\boldsymbol{z} - 1$ [3].

Integrated Parameter Selection Strategy

- Compared to Card-MVO, SVM-MVO introduces additional parameters $(C, q, \epsilon, M, U_{\|w\|_\infty})$. We develop a parameter selection strategy to respect the following principles:

Principle #1: Degenerate solutions ($\boldsymbol{w} = \mathbf{0}$) are useless. We select C and q to avoid degenerate solutions.
Principle #2: big-Ms should not exclude any optimal solutions. We select M and $U_{\|\boldsymbol{w}\|_\infty}$ to avoid excluding optimal solutions.
Principle #3: the risk of any portfolio obtained from SVM-MVO should be within $1 + \kappa$ of the risk of any portfolio obtained from Card-MVO. Given κ, we select ϵ such that any $\boldsymbol{x}$ obtained from SVM-MVO satisfies $\boldsymbol{x}^\intercal \boldsymbol{\Sigma} \boldsymbol{x} \leq (1 + \kappa) \ \boldsymbol{x}_{\text{CardMVO}} \boldsymbol{\Sigma} \boldsymbol{x}_{\text{CardMVO}}$.

- Principles 1-3 $\implies$ the features $\{\boldsymbol{y}^{(i)}\}_{i=1}^N$ and the excess risk tolerance κ are the only additional parameters required for SVM-MVO (in addition to those required by Card-MVO).

Experimental Results (In-Sample)

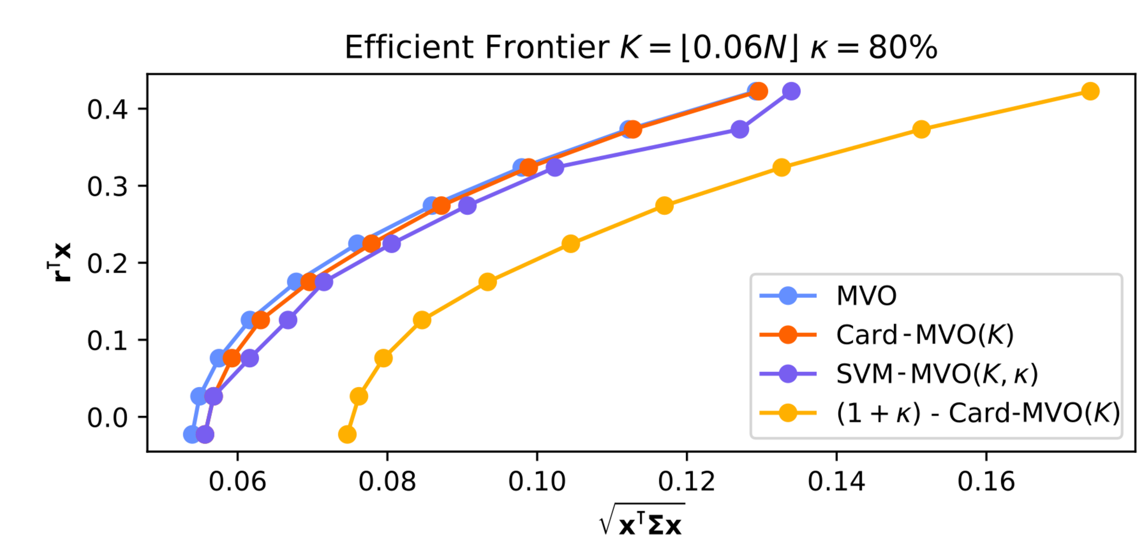

Figure 2. Efficient Frontier for MVO, Card-MVO, and the SVM-MVO model. ϵ is set so that the risk of any SVM-MVO solution is within $(1 + \kappa)$ of Card-MVO's risk.

Experimental Results (Out-of-Sample)

- We consider two datasets to test the SVM-MVO methodology: one of stocks from the S&P 500 and one dataset of exchange-traded funds (ETFs). Each dataset also has an associated feature space $\mathcal{Y}$ to describe the assets at each re-balancing point. For both datasets, we construct sixty-five technical analysis indicators.
- A rolling time window backtest with semi-annual rebalancing is performed. Turnover constraints of the form $\|\boldsymbol{x} - \boldsymbol{x}_0\|_1 \leq C_0$ are added in all the models and periods where $\boldsymbol{x}_0$ denotes the portfolio from the previous period. Figure 3 shows results on the S&P 500 dataset.

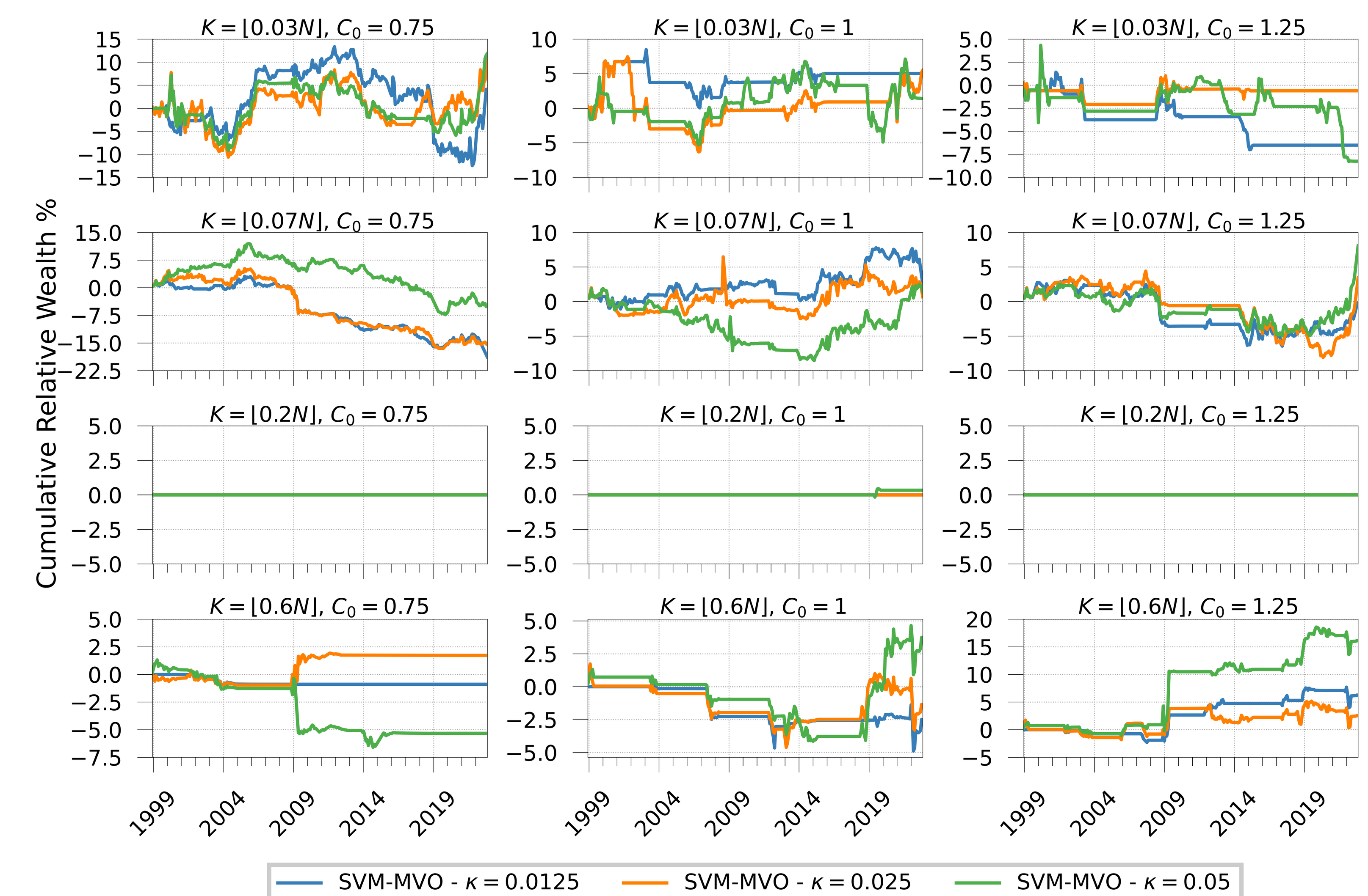

Figure 3. Wealth relative to Card-MVO by turnover limit C_0 and cardinality limit K.

Conclusion

- **A new portfolio selection model**:
  - Augments cardinality-constrained optimization with a preference for portfolios where a low-dimensional hyperplane separates eligible and ineligible assets
  - Presents convex mixed-integer quadratic programming models for joint portfolio and hyperplane selection that are amenable to commercial solvers.
- **A principled parameter selection strategy**: Ensures valid big-M values, risk control, and informative hyperplanes.
- **Empirical demonstration of financial performance**: Results show improved out-of-sample risk-adjusted returns compared to cardinality-constrained mean-variance optimization.

Acknowledgements

The authors were supported by NSERC Grant 455963 and by Yonsei Business Research Institute.

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